Chapter 9 : Integration :

Exercise 9 j

(1) (a) let $I = \int \sin^4 \theta \, d\theta$

$$= \int \left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta$$

$$= \frac{1}{4}\int 1 - 2\cos 2\theta + \cos^2 2\theta \, d\theta$$

$$= \frac{1}{4}\int 1 - 2\cos 2\theta + \frac{1}{2}(\cos 4\theta + 1)\, d\theta$$

$$= \frac{1}{4}\theta - \frac{1}{4}\sin 2\theta + \frac{1}{8}\cdot\frac{1}{4}\sin 4\theta + \frac{1}{8}\theta + c$$

$$= \frac{3}{8}\theta - \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta + c$$

(b) let $I = \int \cos^3\theta \, d\theta = \int \cos\theta\,(\cos^2\theta)\, d\theta$

$$= \int \cos\theta\,(1-\sin^2\theta)\, d\theta = \int \cos\theta - \cos\theta . \sin^2\theta \, d\theta$$

$$= \sin\theta - \int \cos\theta . \sin^2\theta \, d\theta$$

let $u = \sin\theta$ & continue, to get $I = \sin\theta - \frac{1}{3}\sin^3\theta + c$

(c) let $I = \int \tan^4\theta \, d\theta = \int \tan^2\theta \cdot \tan^2\theta \, d\theta$

$$= \int \tan^2\theta \, (\sec^2\theta - 1) \, d\theta$$

$$= \int \tan^2\theta \cdot \sec^2\theta - \tan^2\theta \, d\theta$$

$$= \int \tan^2\theta \cdot \sec^2\theta - (\sec^2\theta - 1) \, d\theta$$

$$= \int \tan^2\theta \cdot \sec^2\theta - \sec^2\theta + 1 \, d\theta$$

$$= \int \tan^2\theta \cdot \sec^2\theta \, d\theta - \tan\theta + \theta + c$$

let $u = \tan\theta$ & continue, to get

$$I = \frac{1}{3}\tan^3\theta - \tan\theta + \theta + c$$

(d) let $I = \int \sin^3\theta \, d\theta = \int \sin\theta \cdot \sin^2\theta \, d\theta$

$$= \int \sin\theta \cdot (1 - \cos^2\theta) \, d\theta$$

$$= \int \sin\theta - \sin\theta \cdot \cos^2\theta \, d\theta$$

$$= -\cos\theta - \int \sin\theta \cdot \cos^2\theta \, d\theta$$

let $u = \cos\theta$ & continue, to get

$$I = -\cos\theta + \frac{1}{3}\cos^3\theta + c$$

(e) let $I = \int \cos^4\theta \, d\theta$

$= \int (\cos^2\theta)^2 \, d\theta$

$= \int \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$, by $\cos 2\theta = 2\cos^2\theta - 1$

So $I = \frac{1}{4}\int 1 + 2\cos 2\theta + \cos^2 2\theta \, d\theta$

$= \frac{1}{4}\int 1 + 2\cos 2\theta + \left(\frac{1+\cos 4\theta}{2}\right) d\theta$

$= \frac{1}{4}\int 1 + 2\cos 2\theta + \frac{1}{2} + \frac{1}{2}\cos 4\theta \, d\theta$

$= \frac{1}{4}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\theta + \frac{1}{32}\sin 4\theta + c$

$= \frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta + c$

(f) let $I = \int \tan^5\theta \, d\theta = \int \tan^3\theta . \tan^2\theta \, d\theta$

$= \int \tan^3\theta . (1 - \sec^2\theta) d\theta$

$= \int \tan^3\theta - \tan^3\theta . \sec^2\theta \, d\theta$

$= \int \tan\theta . \tan^2\theta - \tan^3\theta . \sec^2\theta \, d\theta$

$= \int \tan\theta . (1 - \sec^2\theta) - \tan^3\theta \sec^2\theta \, d\theta$

$= \int \tan\theta - \sec^2\theta . \tan\theta - \tan^3\theta . \sec^2\theta \, d\theta$

$= -\ln|\cos\theta| - \int \sec^2\theta . \tan\theta + \tan^3\theta . \sec^2\theta \, d\theta$

let $u = \tan\theta$ & continue, to get

$$I = -\ln|\cos\theta| - \frac{1}{2}\tan^2\theta - \frac{1}{4}\tan^4\theta + c$$

(2) let $I = \int 2\sin 4\theta \cos 3\theta \, d\theta$

(a)

Note: $\sin A + \sin B = 2\sin\frac{A+B}{2} \cdot \cos\frac{A-B}{2}$

So $4\theta = \frac{A+B}{2}$ & $3\theta = \frac{A-B}{2} \Rightarrow A = 7\theta,\ B = \theta$

So $I = \int \sin 7\theta + \sin\theta \, d\theta = -\frac{1}{7}\cos 7\theta - \cos\theta + C$

(b) $I = \int 2\cos 2\theta \cos 5\theta \, d\theta$

Note: $\cos A + \cos B = 2 \cdot \cos\frac{A+B}{2} \cdot \cos\frac{A-B}{2}$

So $\frac{A+B}{2} = 5\theta$ & $\frac{A-B}{2} = 2\theta \Rightarrow A = 7\theta,\ B = 3\theta$

So $I = \int \cos 7\theta + \cos 3\theta \, d\theta = \frac{1}{7}\sin 7\theta + \frac{1}{3}\sin 3\theta + c$

(c) $I = \int \sin 2\theta \cos 6\theta \, d\theta = \frac{1}{2} \int 2 \sin 2\theta \cos 6\theta \, d\theta$

Note: $\sin A - \sin B = 2 \sin \frac{A-B}{2} \cdot \cos \frac{A+B}{2}$

So $\frac{A-B}{2} = 2\theta$ & $\frac{A+B}{2} = 6\theta \Rightarrow A = 8\theta, \; B = 4\theta$

So $I = \frac{1}{2} \int \sin 8\theta - \sin 4\theta \, d\theta = -\frac{1}{16} \cos 8\theta + \frac{1}{8} \cos 4\theta + C$

(d) Let $I = \int \sin\theta \sin 3\theta \, d\theta = -\frac{1}{2} \int -2 \sin\theta \sin 3\theta \, d\theta$

Note: $\cos A - \cos B = -2 . \sin \frac{A+B}{2} \cdot \sin \frac{A-B}{2}$

So $3\theta = \frac{A+B}{2}$; $\theta = \frac{A-B}{2} \Rightarrow A = 4\theta, \; B = 2\theta$

So $I = -\frac{1}{2} \int \cos 4\theta - \cos 2\theta \, d\theta$

$= -\frac{1}{8} \sin 4\theta + \frac{1}{4} \sin 2\theta + C$

(e) let $I = \int 2 \sin nx \cdot \cos mx \, dx$

Note: $\sin A + \sin B = 2 \sin \frac{A+B}{2} \cdot \cos \frac{A-B}{2}$

So $nx = \frac{A+B}{2}$ & $mx = \frac{A-B}{2}$

$\Rightarrow A = (n+m)x, \; B = (n-m)x$

$\therefore \quad I = \int \sin(n+m)x + \sin(n-m)x \, dx$

$= -\frac{1}{n+m} \cdot \cos(n+m)x - \frac{1}{n-m} \cdot \cos(n-m)x + C$

(f) Let $I = \int 2\cos\frac{u}{2} \cdot \cos\frac{u}{3} \, du$

Note : $\cos A + \cos B = 2\cos\frac{A+B}{2} \cdot \cos\frac{A-B}{2}$

So $\frac{u}{2} = \frac{A+B}{2}$ & $\frac{u}{3} = \frac{A-B}{2}$.

$\Rightarrow A = \frac{5}{6}u \; , \; B = \frac{1}{6}u$

So $I = \int \cos\frac{5}{6}u + \cos\frac{1}{6}u \, du$

$= \frac{6}{5}\sin\frac{5}{6}u + 6\sin\frac{1}{6}u + C$

(g) Let $I = \int \cos nx \cdot \cos mx \, dx = \frac{1}{2}\int 2\cos nx \cdot \cos mx \, dx$

Note : $\cos A + \cos B = 2\cos\frac{A+B}{2} \cdot \cos\frac{A-B}{2}$

So $nx = \frac{A+B}{2}$ & $mx = \frac{A-B}{2}$

$\Rightarrow A = (n+m)x$ & $B = (n-m)x$

$$\therefore \quad I = \frac{1}{2}\int \cos(n+m)x + \cos(n-m)x \, dx$$

$$= \frac{1}{2(n+m)} \cdot \sin(n+m)x + \frac{1}{2(n-m)} \cdot \sin(n-m)x + c$$

3 a let $I = \int \sin^2 x . \cos^3 x \, dx$

$$= \int \sin^2 x . \cos x . \cos^2 x \, dx$$

$$= \int \sin^2 x . \cos x . (1 - \sin^2 x) \, dx$$

$$= \int \sin^2 x . \cos x - \sin^4 x . \cos x \, dx$$

let $u = \sin x$ & continue, to get

$$I = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + c$$

b Same as a : let $I = \int \sin^{10} x . \cos^3 x \, dx$

$$= \int \sin^{10} x . \cos^2 x . \cos x \, dx$$

$$= \int \sin^{10} x (1 - \sin^2 x) . \cos x \, dx$$

$$= \int \sin^{10} x . \cos x - \sin^{12} x . \cos x \, dx$$

let $u = \sin x$ & continue, to get

$$I = \frac{1}{11}\sin^{11} x - \frac{1}{13}\sin^{13} x + c.$$

(c) let $I = \int \sin^2 x \cdot \cos^2 x \, dx$

$$= \int (\sin x \cdot \cos x)^2 \, dx$$

$$= \int \left(\tfrac{1}{2} \cdot \sin 2x\right)^2 \, dx$$

$$= \frac{1}{4} \int \sin^2 2x \, dx$$

By $\cos 2x = 1 - 2\sin^2 x$ we have $\sin^2 x = \dfrac{1 - \cos 2x}{2}$

$\therefore$

$$I = \frac{1}{4} \int \frac{1 - \cos 4x}{2} \, dx$$

$$= \frac{1}{8}\left(x - \frac{1}{4}\sin 4x\right) + c$$

$$= \frac{1}{32}\left(4x - \sin 4x\right) + c$$

(d) let $I = \int \tan^2 x \sec^4 x \, dx$

$$= \int \tan^2 x \cdot \sec^2 x \cdot \sec^2 x \, dx$$

$$= \int \tan^2 x \left(1 + \tan^2 x\right) \sec^2 x \, dx$$

$$= \int \tan^2 x \cdot \sec^2 x + \tan^4 x \cdot \sec^2 x \, dx$$

let $u = \tan x$ & continue, to get

$$I = \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + c$$

(e) let $I = \int \sin^3 2x \cdot \cos^2 2x \, dx$

$$= \int \sin 2x \cdot \sin^2 2x \cdot \cos^2 2x \, dx$$

$$= \int \sin 2x \left(1 - \cos^2 2x\right) \cos^2 2x \, dx$$

$$= \int \sin 2x \cos^2 2x - \sin 2x \cos^4 2x \, dx$$

$$= -\frac{1}{2}\cdot\frac{1}{3}\cos^3 2x + \frac{1}{2}\cdot\frac{1}{5}\cos^5 2x + c$$

$$= \frac{1}{10}\cos^5 2x - \frac{1}{6}\cos^3 2x + c$$

(f) let $I = \int \sin^n x \cdot \cos^3 x \, dx$ (Same type of Question as (a))

$$= \int \sin^n x \cdot \cos x \cdot \cos^2 x \, dx$$

$$= \int \sin^n x \cdot \cos x \cdot \left(1 - \sin^2 x\right) dx$$

$$= \int \sin^n x \cdot \cos x - \sin^{n+2} x \cdot \cos x \, dx$$

$$= \frac{1}{n+1}\cdot \sin^{n+1} x - \frac{1}{n+3}\cdot \sin^{n+3} x + c$$

(g) let $I = \int \frac{\cos^2 x}{\operatorname{cosec}^3 x}\, dx = \int \cos^2 x \cdot \sin^3 x\, dx$

$= \int \cos^2 x \cdot \sin x \cdot \sin^2 x\, dx$

$= \int \cos^2 x \cdot \sin x \cdot (1 - \cos^2 x)\, dx$

$= \int \cos^2 x \cdot \sin x - \cos^4 x \cdot \sin x\, dx$

$= -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + c$

(h) let $I = \int \frac{\tan^3 x}{\cos^2 x}\, dx = \int \frac{\sin^3 x}{\cos^5 x}\, dx$

OR: $I = \int \tan^3 x \cdot \sec^2 x\, dx$ ← use This one.

$= \frac{1}{4}\tan^4 x + c$